Practice Exam 2 Solutions

This is a collection of relevant problems I have given on previous exams; it does not correspond exactly to a one-hour test. This exam is closed-book and closed-note. Please take some time to check your work. If you need extra space, write on the back. You must show your work to receive any partial credit.

1. Given the sets

$$A = \{a, i, u\}$$
$$B = \{i, o, u\},$$

list the elements of each of the following sets:

(a) $A \cup B$

$\{a, i, o, u\}$

(b) $A \cap B$

$\{i, u\}$

(c) $A - B$

$\{a\}$

(d) $(A - B) \cup (B - A)$

$\{a, o\}$

(e) $A \times B$

$\{(a,i), (a,o), (a,u), (i,i), (i,o), (i,u), (u,i), (u,o), (u,u)\}$

(f) $P(A)$

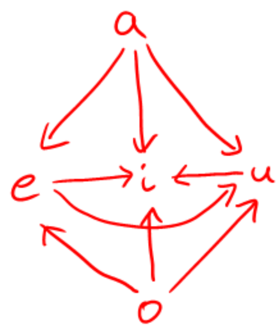
$\{\phi, \{a\}, \{i\}, \{u\}, \{a,i\}, \{a,u\}, \{i,u\}, \{a,i,u\}\}$

2. Let A be the set $\{a, e, i, o, u\}$, and consider the relation R on A whose graph is given by the following adjacency matrix:

	a	e	i	o	u
a	0	1	1	0	1
e	0	0	1	0	1
i	0	0	0	0	0
o	0	1	1	0	1
u	0	0	1	0	0

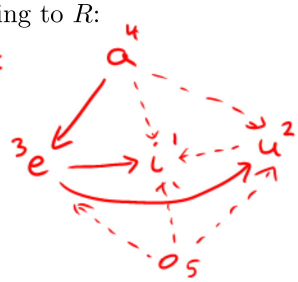
(Recall that the convention is that the cell at row x, column y is 1 if $x \, R \, y$.)

(a) Draw the graph of R:



(b) Either identify a cycle in the graph of R, or give a topological ordering of the elements of A according to R:

dfs:



none are back edges, so acyclic

a topo. ordering is

o, a, e, u, i

(the only other is a, o, e, u, i)

(c) Which of the following properties apply to R: reflexive, symmetric, transitive, antisymmetric?

- not reflexive or symmetric.
  (a $\not R$ a) (a R e but e $\not R$ a)
- transitive — applying Warshall's algorithm and considering each node as a pivot doesn't add any edges.
- antisymmetric — there is no x, y such that x R y and y R x.
  (not even for x=y)

3. Given the set $S = \{A, B, C, D, E, F, G\}$, we may represent any subset of S by its characteristic vector, which will have seven bits. For example, the set $\{A, B, D\}$ is represented by the bit vector `1101000`. Consider these named subsets of S:

$$I = \{C, E, G\} \qquad iii = \{E, G, B\} \qquad V = \{G, B, D\}$$

0010101 0100101 0101001

Give the bit vector representation for each of the following:

(a) $I \cup iii$ = $\{B, C, E, G\}$

0110101

$$\begin{array}{rr} & 0010101 \\ \text{OR} & 0100101 \\ \hline & 0110101 \end{array}$$

(b) $I \cap iii$ = $\{E, G\}$

0000101

$$\begin{array}{rr} & 0010101 \\ \text{AND} & 0100101 \\ \hline & 0000101 \end{array}$$

(c) $I - V$ = $\{C, E\}$

0010100

$$\begin{array}{rr} & 0010101 \\ \text{AND} & 1010110 \\ \hline & 0010100 \end{array}$$

$1010110 = \overline{V}$ (complement)

(d) $(I - V) \cup (V - I)$ $= \{C, E\} \cup \{B, D\}$

$= \{B, C, D, E\}$

0111100

$$\begin{array}{rr} & 0010101 \\ \text{XOR} & 0101001 \\ \hline & 0111100 \end{array}$$

(e) $(I \cup V) - (I \cap V)$ $= \{B, C, D, E, G\} - \{G\}$

$= \{B, C, D, E\}$

same as (d)

4. For the same set S, how many elements are in the powerset, $P(S)$? What is the set of bit vectors corresponding to the elements of $P(S)$ (give a simple description)?

$|S| = 7$, so $|P(S)| = 2^7 = \underline{128}$

the bit vectors are the 128 7-bit binary numbers from 0000000 (0) to 1111111 (127).

5. We observed in class that the logical implication operator, $\rightarrow$, behaves like a transitive relation. Consider the set $\mathcal{E}$ of all logical expressions. If E and F are elements of $\mathcal{E}$, then we will define the relation $\Rightarrow$ on $\mathcal{E}$ by saying that $E \Rightarrow F$ exactly when the expression $E \rightarrow F$ is a tautology. For example, $(p+q) \Rightarrow (q+p)$. However, $(p+q) \not\Rightarrow pq$, because when p is true and q is false, the condition $p+q$ is true but the conclusion pq is not.

   (a) Doe $pq \Rightarrow (p+q)$ hold?

   yes — whenever pq (p AND q) is true, it is also the case that $p+q$ (p OR q) is true.

   (b) Is $\Rightarrow$ reflexive? Why or why not?

   yes — for every E, $E \rightarrow E$ is a tautology

   (c) Is $\Rightarrow$ symmetric? Why or why not?

   no — by the above, $pq \Rightarrow p+q$ but $p+q \not\Rightarrow pq$

   (d) Is $\Rightarrow$ antisymmetric? Why or why not?

   no — for example, both $p+q \Rightarrow q+p$ and $q+p \Rightarrow p+q$, but $p+q \neq q+p$ (as expressions; they are however logically equivalent)

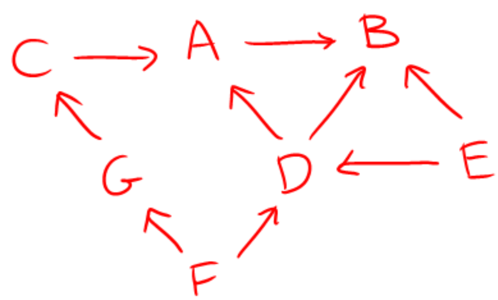
   (e) Is $\Rightarrow$ transitive? Why or why not?

   yes — if $E \rightarrow F$ and $F \rightarrow G$ are tautologies, then so is $E \rightarrow G$, for all E, F, G.

6. Here is an adjacency list representation of a directed graph:

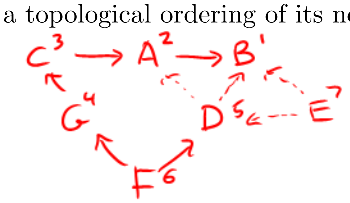
Node	Successors
A	B
B	(none)
C	A
D	A, B
E	B, D
F	D, G
G	C

(a) Draw the graph.



(b) If the graph is acyclic, give a topological ordering of its nodes; otherwise, identify a cycle.

dfs starting at F:



no back edges, so acyclic

topo. ordering: (one of several)

E, F, D, G, C, A, B

(c) What is the longest path in the graph that never revisits a node?

(by inspection)

$F \to G \to C \to A \to B$

(d) Define the distance between two vertices as the length of the shortest path between them, or ∞ if there is no such path. What is the greatest non-infinite distance in this graph?

3: G to B

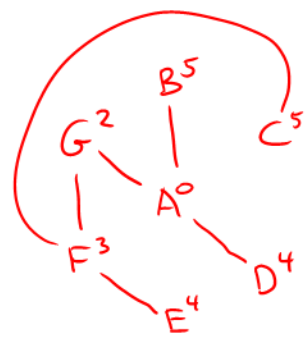
7. Here is an adjacency matrix representation of an undirected weighted graph (a weight of ∞ means there is no edge between those vertices):

	A	B	C	D	E	F	G
A	0	5	7	4	6	5	2
B	5	0	1	∞	∞	∞	∞
C	7	1	0	8	∞	2	∞
D	4	∞	8	0	1	∞	∞
E	6	∞	∞	1	0	1	∞
F	5	∞	2	∞	1	0	1
G	2	∞	∞	∞	∞	1	0

(a) Draw the graph. (omitting self-loops)

(b) Find the (weighted) shortest path from A to E.

using Dijkstra's algorithm:



(c) Would the answer to the previous question change if the weight of the edge between B and C were changed to -1? Why or why not?

no change — the shortest path from A to either B or C is 5 units, so the -1 between B & C couldn't make a path shorter than 4 (in fact, the best path using that edge is A–B–C–F–E of weight 7)

(d) Find a minimum spanning tree for the graph.

This uses Prim's algorithm (like Dijkstra w/ different priority):